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Deformations in Mathematics and Physics

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Abstract This text is my introductory talk given at the Workshop "Deformations and Contractions in Mathematics and Physics" in Oberwolfach in January 2006.

1 Introduction

Deforming a given mathematical structure is a tool of fundamental importance in most parts of mathematics, mathematical physics and physics. Contractions have been developed mainly by physicists. The aim of this workshop is to bring together experts in these complementary topics. Deformations and contractions have been investigated by researchers who had different approaches and goals. Tools such as cohomology, gradings etc. which are utilized in the study of one concept, are likely to be useful for the other concept as well. At this meeting there are mathematicians, mathematical physicists and physicists as well. It seemed to us that such a meeting would benefit all.

1.1 The notion of deformation

The theory of deformations originated with the problem of classifying all possible pairwise non-isomorphic complex structures on a given differentiable real manifold. The fundamental idea, which should be credited to Riemann, was to introduce an analytic structure therein. The notion of local and infinitesimal deformations of a complex analytic manifold first appeared in the work of Kodaira and Spencer [1, 2]. In particular, they proved that infinitesimal deformations can be parametrized by the corresponding cohomology group. The deformation theory of compact complex manifolds was devised by Kuranishi [3] and Palamodov [4]. Shortly after the work of Kodaira and Spencer, algebro-geometric foundations were systematically developed by Artin [5] and Schlessinger [6]. Formal deformations of arbitrary rings

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and associative algebras, and the related cohomology questions, were first investigated by Gerstenhaber, in a series of articles [7–10]. The notion of deformation was applied to Lie algebras by Nijenhuis and Richardson [11, 12].

Because various fields in mathematics and physics exist in which deformations are used, we focused on the topic of the conference. We mainly consider here deformations of algebras (in particular, of Lie algebras), groups, and related algebraic structures and their applications to problems in physics. Beside the central topic, to open up fertile interaction, we invited also researchers from neighboring disciplines. One such topic with tight interaction is deformation quantization. But there will also be others, like quantum groups, deformation of Hopf algebras, Leibniz and dialgebras, infinity algebras, q-deformed physics, fuzzy spaces, quantum systems as deformations of classical systems, etc.

Deformation is one of the tools used to study a specific object, by deforming it into some families of "similar" structure objects. This way we get a richer picture about the original object itself. But there is also another question approached via deformation. Roughly speaking, it is the question, can we equip the set of mathematical structures under consideration (may be up to certain equivalence) with the structure of a topological or geometric space. In other words, does there exist a moduli space for these structures. If so, then for a fixed object deformations of this object should reflect the local structure of the moduli space at the point corresponding to this object.

Let me give an example: The classification of complex analytic structures or a fixed topological manifold is completely understood. Also in algebraic geometry one has well-developed results in this direction. One of these results is that the local situation at a point $[C]$ of the moduli space is completely governed by the cohomological properties of the geometric object C. As typical example recall that for the moduli space M_g of smooth projective curves of genus g over $\mathbb{C}$ (or, equivalently, compact Riemann surfaces of genus g) the tangent space $T_{[C]}M_g$ can be naturally identified with $H^1(C, T_C)$, where T_C is the sheaf of holomorphic vector fields over C. This extends to higher dimension. In particular, it turns out that for compact complex manifolds M, the condition $H^1(M, T_M) = 0$ implies that M is rigid [13]. Rigidity means that any differentiable family $\pi : M \to B \subset \mathbb{R}$, $0 \in B$ which contains M as the special member, $M_0 := \pi^{-1}(0)$ is trivial in a neighborhood of 0, i.e. for t small enough, $M_t := \pi^{-1}(t) \cong M$. Even more generally, for M a compact complex manifold and $H^1(M, T_M) \neq 0$ there exists a versal family which can be realized locally as a family over a certain subspace of $H^1(M, T_M)$ such that every appearing deformation family is "contained" in this versal family [14].

2 Definitions

For simplicity, consider the Lie algebra case. Let $\mathcal{L}$ be a Lie algebra with Lie bracket μ_0 over a field $\mathbb{K}$.

(a) *Intuitive definition*. A deformation of $\mathcal{L}$ is a one-parameter family $\mathcal{L}_t$ of Lie algebras with the bracket

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \cdots$$

where φ_i are $\mathcal{L}$-valued 2-cochains, i.e. elements of $\mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathcal{L}, \mathcal{L}) = C^2(\mathcal{L}; \mathcal{L})$, and $\mathcal{L}_t$ is a Lie algebra for each $t \in \mathbb{K}$. Two deformations, $\mathcal{L}_t$ and $\mathcal{L}'_t$ are equivalent if there exists a linear automorphism $\widehat{\psi}_t = \mathrm{id} + \psi_1 t + \psi_2 t^2 + \cdots$ of $\mathcal{L}$ where ψ_i are linear maps over $\mathbb{K}$, i.e. elements of $C^1(\mathcal{L}, \mathcal{L})$ such that

$$\mu'_t(x, y) = \widehat{\psi}_t^{-1}(\mu_t(\widehat{\psi}_t(x), \widehat{\psi}_t(y))) \quad \text{for } x, y \in \mathcal{L}.$$

The Jacobi identity for the algebras $\mathcal{L}_t$ implies that the 2-cochain φ_1 is indeed a cocycle, i.e. $d_2\varphi_1 = 0$. If φ_1 vanishes identically, the first nonvanishing φ_i will be a cocycle. If μ'_t is an equivalent deformation with cochains φ'_i, then

$$\varphi'_1 - \varphi_1 = d_1\psi_1,$$

hence every equivalence class of deformations defines uniquely an element of $H^2(\mathcal{L}, \mathcal{L})$ (see [7–10]).

(b) *General definition*. Consider now a deformation $\mathcal{L}_t$ not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra over $\mathbb{K}$ with identity as base of a deformation. Let us fix an augmentation $\varepsilon : A \to \mathbb{K}$, $\varepsilon(1) = 1$, and set $\operatorname{Ker}\varepsilon = m$, which is a maximal ideal.

Definition A deformation λ of $\mathcal{L}$ with base (A, m) is a Lie A-algebra structure on the tensor product $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[\ ,\ ]_\lambda$ such that

$$\varepsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called equivalent (or isomorphic) if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\varepsilon \otimes \mathrm{id}$.

A deformation with base A is called *local* if the algebra A is local, and it is called *infinitesimal* if, in addition to this, $m^2 = 0$. For general commutative algebra base, we call the deformation *global*.

(c) *Formal deformations*. Let A be a complete local algebra (completeness means that $A = \overleftarrow{\lim}_{n\to\infty}(A/m^n)$, where m is the maximal ideal in A). A formal deformation of $\mathcal{L}$ with base A is a Lie A-algebra structure on the completed tensor product $A\widehat{\otimes}\mathcal{L} = \overleftarrow{\lim}_{n\to\infty}((A/m^n) \otimes \mathcal{L})$ s.t.

$$\varepsilon\widehat{\otimes}\mathrm{id} : A\widehat{\otimes}\mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

The previous notion of equivalence can be extended to formal deformations in an obvious way.

(d) *Formal versal deformations*. It is known that in the category of algebraic varieties the quotient by a group action does not always exist [15]. Specifically, there is no universal deformation in general of a Lie algebra $\mathcal{L}$ with a commutative algebra base A with the property that for any other deformation of $\mathcal{L}$ with base B there exists a unique homomorphism $f : B \to A$ that induces an equivalent deformation. If such a homomorphism exists (but not unique), we call the deformation of $\mathcal{L}$ with base A *versal*.

The classical one-parameter deformation theory did not study the versal property of deformations. A more general deformation theory of Lie algebras follows from Schlessinger's work [6]. Namely, for complete local algebra base deformations, under some minor restriction, there exists a so-called miniversal deformation:

A formal deformation η of a Lie algebra $\mathcal{L}$ with a complete local algebra base B is called *miniversal*, if

(i) for any formal deformation λ of $\mathcal{L}$ with any complete local base A there exists a homomorphism $f : B \to A$ s.t. the deformation λ is equivalent to the push-out of η by f;
(ii) if A satisfies $m^2 = 0$, then f is unique (see Refs. [16–18]).

Theorem *Let $H^2(\mathcal{L}, \mathcal{L})$ be finite-dimensional. Then there exists a versal formal deformation of $\mathcal{L}$, and the base of this versal deformation is formally embedded into $H^2(\mathcal{L}, \mathcal{L})$, i.e. it can be described in $H^2(\mathcal{L}, \mathcal{L})$ by a finite system of formal equations.*

Another question is how to construct such a deformation. Fuchs and I gave a construction which can be carried out with a computer (see Ref. [19]).

The situation is much more complicated for global deformations, where we loose the cohomology as a tool for obtaining deformations and so far there is no way to get a versal object.

(e) *Rigidity*. A Lie algebra $\mathcal{L}$ is called *rigid* if every deformation is equivalent to a trivial deformation. There are various notions of rigidity: infinitesimal, formal, geometric, analytic, global, etc.

Proposition *$\mathcal{L}$ is infinitesimally rigid if and only if $H^2(\mathcal{L}, \mathcal{L}) = 0$.*

Theorem *If $\mathcal{L}$ is finite-dimensional and $H^2(\mathcal{L}, \mathcal{L}) = 0$ then $\mathcal{L}$ is rigid in every sense.*

Theorem *Let $\mathcal{L}$ be an arbitrary Lie algebra. If $H^2(\mathcal{L}, \mathcal{L}) = 0$, then $\mathcal{L}$ is formally rigid.*

Remark The above formal rigidity assertion is *not* true for global deformations. As an example, let us mention the Virasoro algebra, for which the space $H^2(\mathcal{L}, \mathcal{L}) = 0$, so it is formally rigid, but it has lot's of nice nontrivial global deformations, like the Krichever–Novikov type algebras.

3 Applications

(a) Global deformations of Witt, Virasoro and affine Kac–Moody algebras are important in the theory of 2-dimensional conformal fields and their quantization. In the case of higher genus Riemann surfaces Krichever and Novikov proposed the use of global operator fields which are given with the help of the Lie algebra of vector fields of Krichever–Novikov type, certain related algebras, and their representations. Deformations of affine Lie algebras are used in the global operator approach to the Wess–Zumino–Witten–Novikov models appearing in the quantization of conformal field theory (see Refs. [20, 21]).

(b) Deformation quantization. In 1978 Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer opened up a systematic approach to quantization via the deformation theory of algebras. A *deformation quantization* (or a *star product*) is an associative deformation of the algebra of classical observables in the direction of the Poisson bracket. Nowadays we have complete existence and classification results on the formal level. In 1997 Kontsevich proved the "formality conjecture" which implies that any finite-dimensional Poisson manifold can be quantized (in the sense of deformation quantization). There are a lot of open questions around deformation quantization like the treatment of infinite dimensional spaces, the question of existence of a dense subalgebra of functions such that the star product of these elements is convergent, invariant star product, its relation to physics or noncommutative spaces, specific examples in physics, etc.

(c) Recently, much attention has been paid to deformations of homotopy algebras, operands, Leibniz algebras, dialgebras, flag varieties, symplectic algebroids, etc., and to more general derived deformation theory.

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